A comparison of construction techniques for modular fusion categories

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Quantum Groups Seminar

January 24, 2023

WHY MTC'S?

- THEY APPEAR IN MANY AREAS OF MATH AND HAVE APPLICATIONS IN PHYSICS.
- NATURAL HOSTS OF QUANTUM SYMMETRIES.

WHAT ABOUT THEM?

- IT IS A "YOUNG" THEORY ...

WE NEED MORE EXAMPLES.

SOME PROBLEMS I LIKE TO THINK ABOUT:

- CLASSIFICATION
- PROPERTIES / INVARIANTS
- CONSTRUCTIONS

DEFINITION ... BY EXAMPLE (of fusion CATEGORY)

$\mathcal{C} = \mathrm{REP}(G)$ = finite dimensional representations of G over k ($k = \bar{k}$, char $k = 0$)

GIVEN $V, W \in \mathrm{REP}(G)$:

- $\mathrm{Hom}_{\mathcal{C}}(V, W)$ = intertwiners → k-v.s.
- $V \otimes W \in \mathrm{REP}(G)$ → $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$
- $k \in \mathrm{REP}(G)$ → $g \cdot 1 = 1$
- $V^* \in \mathrm{REP}(G)$, $T \in V^*$ → $(\underbrace{g \cdot T}_{\in V^*})(v) = T(g^{-1} \cdot v)$

  $V^* = \mathrm{Lin}(V, k)$
- $\tau : V \otimes W \to W \otimes V \in \mathrm{Hom}_{\mathcal{C}}(V \otimes W, W \otimes V)$

  $v \otimes w \mapsto w \otimes v$

DEF.: THE CATEGORY $\mathcal{C}$ is A **fusion CATEGORY** over k if:

- $\mathcal{C}$ is <u>ABELIAN k-LINEAR</u>: $\oplus$, ker, coker, $\underset{\mathcal{C}}{\mathrm{Hom}}(X,Y) \downarrow$ k-v.s.
- $\mathcal{C}$ is <u>MONOIDAL</u>: $(\otimes, \alpha, \mathbb{1}, \ell, r)$ + ⬠ + $\triangle$ AX.
- $\mathcal{C}$ is <u>RIGID</u>: $\forall X \in \mathcal{C}\ \exists\, (\underset{\in \mathcal{C}}{X^*}, ev_X : X^* \otimes X \to \mathbb{1}, coev_X : \mathbb{1} \to X \otimes X^*)$ + ZIG-ZAG AX.

- $\mathcal{C}$ is <u>SEMISIMPLE</u>: $X = \bigoplus X_i \leftarrow$ simples
  $\hookrightarrow$ finite sum
- $\mathcal{C}$ is <u>FINITE</u>: fin. many iso classes of simples.
- $\mathbb{1}$ simple

WE SAY THAT $\mathcal{C}$ is **BRAIDED** if IT IS EQUIPPED WITH NAT. isom. $\sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ + ⬡$_s$ AX.

- $\mathcal{C}$ strict $\begin{pmatrix} \alpha = id \\ \ell = r = id \end{pmatrix}$ $\sigma_{X\otimes Y, Z} = (\sigma_{X,Z} \otimes id) \circ (id \otimes \sigma_{Y,Z})$

* PREMODULAR (RIBBON) + NON-DEG.

NOTATION:

- $\mathrm{Irr}(\mathcal{C}) = \{ X \text{ simple in } \mathcal{C} \} / \sim \; = \{ X_0 = \mathbb{1}, \dots, X_{r-1} \}$

  $\mathrm{Rank}(\mathcal{C}) = |\mathrm{Irr}(\mathcal{C})| = r$

- $\mathrm{Inv}(\mathcal{C}) = \{ X \text{ invertible in } \mathcal{C} \}$ $\quad X \otimes X^* \underset{\text{coev}}{\cong} \mathbb{1} \overset{\text{ev}}{\cong} X^* \otimes X$

- $G(\mathcal{C}) = \mathrm{Inv}(\mathcal{C}) / \sim$ Group of inv.

- fusion rules: $X \otimes Y = \bigoplus_{z \in \mathrm{Irr}(\mathcal{C})} N_{x,y}^{z} \, z$ $\quad N_{x,y}^{z} \in \mathbb{N}_0$

  $x, y \in \mathrm{Irr}(\mathcal{C})$ $\quad \curvearrowright_{\mathcal{C}}$

- fpdim: $L_x = X \otimes - \longrightarrow N_x = \left(N_{x,y}^{z} \right)_{y,z}$

$\mathrm{fpdim}(x) \doteq$ MAX. NON-NEGATIVE EIGENVALUE of N_x

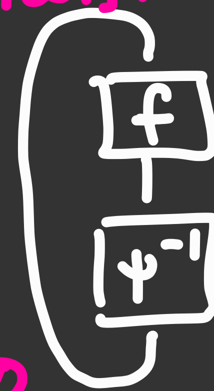
$\in \mathbb{R}_{\geq 0}$

$$\mathrm{fpdim}\, \mathcal{C} = \sum_{x \in \mathrm{Irr}\, \mathcal{C}} (\mathrm{fpdim}\, x)^2$$

Modular Data

Ribbon structure $\rightsquigarrow$ θ twist

$\theta_X : X \xrightarrow{\sim} X$

- Trace: $f \in \mathrm{End}(X) \longrightarrow \mathrm{Tr}(f) =$  Pivotal struct.

  $\in \mathrm{End}(\mathbb{1}) = \Bbbk \, \mathrm{id}_{\mathbb{1}}$

- T-matrix: $\theta_X : X \xrightarrow{\sim} X$ twist.

  $\theta_X \in \mathrm{Aut}(X) \simeq \Bbbk^\times \mathrm{id}$

  $T_{X,Y} = \theta_X \, \delta_{X,Y}, \quad X, Y \in \mathrm{Irr}(\mathcal{C})$

- S-matrix:

  $S_{X,Y} = \mathrm{Tr}\left(\sigma_{Y^*,X} \circ \sigma_{X,Y^*}\right) = \frac{1}{D}$  Hopf link

  $X, Y \in \mathrm{Irr}(\mathcal{C})$

$\mathcal{C}$ premodular

MTC $\longleftrightarrow$ $\det S \neq 0$.

- $\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ \sigma_{Y,X} \circ \sigma_{X,Y}$

$$X \otimes Y \xrightarrow{\sigma_{X,Y}} Y \otimes X \xrightarrow{\sigma_{Y,X}} X \otimes Y$$

- θ_{X^*}

$$\theta \rightsquigarrow \pi$$

EXAMPLES:

1. POINTED MODULAR CATEGORIES: G finite group, & non-deg. quadratic form

VEC_G^q = fin. dim. G-graded v.s.

2. DOUBLES: G fin. group, $\omega \in H^3(G, \mathbb{C}^*)$.

$DG = kG \otimes k^G$ —— $\mathrm{Rep}(D^\omega G)$

3. FIB: $\mathbb{1}, \tau \longrightarrow \frac{1+\sqrt{5}}{2}$

$\tau \otimes \tau = \mathbb{1} \oplus \tau$

4. ISING: $\mathbb{1}, \psi, \sigma$

$\mathbb{1} \to 1, \quad \psi \to 1, \quad \sigma \to \sqrt{2}$

$\sigma \otimes \psi = \sigma = \psi \otimes \sigma$

$\sigma \otimes \sigma = \mathbb{1} \oplus \psi$

5. QUANTUM GROUPS: $\mathfrak{g}$ simple Lie Alg $\to U_q\mathfrak{g}$

(restrictions on ℓ) $\quad q = e^{\pi i/\ell} \quad \to \quad \mathrm{Rep}\, U_q\mathfrak{g}$

NEW FROM OLD ... → CONSTRUCTIONS!

- **DELIGNE PRODUCT:** $\mathcal{C}, \mathcal{D}$ MTCs → DELIGNE PRODUCT $\mathcal{C} \boxtimes \mathcal{D}$ MTC
- **DRINFELD CENTER:** $\mathcal{C}$ "SPHERICAL" FUSION CAT. → $Z(\mathcal{C})$ MTC
- **MODULARIZATION:** [MÜGER - BRUGUIÈRES] " $\mathcal{C}_G$ " MTC

$\mathcal{C}$ BRAIDED, PREMODULAR, $Z_2(\mathcal{C}) \simeq \mathrm{REP}(G)$ TANNAKIAN → $\mathcal{C}_G$ MTC

- **CORE:** $\mathcal{C}$ BRAIDED FUS. CAT., MAXIMAL TANN. SUBC. $\mathrm{REP}(G)$

[DGNO]

$(\mathcal{C}_G)_0$ → MODULAR, → SUPER MODULAR

G-CROSSED BRAIDED CAT.

- **GAUGING:** $\mathcal{C}$ MTC, $G \curvearrowright \mathcal{C}$ → $\mathcal{C}^{\mathrm{GAUGE}(G)}$ MTC

[BBCW]
[CGPW]

↓ OBSTRUCTIONS!

- **ZESTING:** $\mathcal{C}$ MTC, $\mathcal{C} = \bigoplus_{g \in G(\mathcal{C})} \mathcal{C}_g$, $\mathcal{C}_{pt} \subseteq \mathcal{C}_0$ → $\tilde{\mathcal{C}}$ MTC

[DGPRZ]

$$Z_2(\mathcal{C}) = \{ X \in \mathcal{C} \mid \sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y} \ \forall Y \in \mathcal{C} \}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\sigma_{X,Y}"] \arrow[rr, bend right, "\mathrm{id}_{X \otimes Y}"'] & Y \otimes X \arrow[r, "\sigma_{Y,X}"] & X \otimes Y
\end{tikzcd}$$

DELIGNE

SYMM. CAT.

$\mathrm{Rep}(G, z)$

$z \in Z(G)$

$|z| \leq 2$

Rep G $\rightsquigarrow$ k^G Algebras comm. in $\mathcal{C}$

$$\mathrm{Mod}_{\mathcal{C}}(k^G) = \mathcal{C}_G$$

GAUGING: $\mathcal{C}$ MTC, $G \curvearrowright \mathcal{C}$ CAT. ACTION

- **STEP 1**: END EXTENSION THEORY

$$\mathcal{C}^{EXT(G)} = \bigoplus_{g \in G} \mathcal{C}_g \; , \quad \mathcal{C}_e = \mathcal{C}$$

$$\#^{o}$$

G-CROSSED B.f.C. $\rightsquigarrow G \curvearrowright \mathcal{C}^{EXT(G)}$

OBSTRUCTIONS!

- **STEP 2**: G-EQUIVARIANTIZATION OF $\mathcal{C}^{EXT(G)}$

$G \curvearrowright \mathcal{C}^{EXT(G)} \rightsquigarrow$ CAT. OF FIXED POINTS UNDER THIS ACTION $\cong$

$$\mathrm{Rep}(G) \subseteq \mathcal{C}^{GAUGE(G)}$$

Zesting... History & Motivation

2014:

- Classification of MTC's of FPdim = 36 Rank 10 fusion ring similar but it not the same as $SU(3)_3$ $\longrightarrow$ By "twisting" the fusion rules by invertibles

2016:

- Minimal closures of super-modular categories given one MME $\longrightarrow$ $\exists$ 16 of them $\longrightarrow$ 8 of them. Given 1 of them

[DMNO], [KLW], [B+], [Jf.R]

$$PSU(2)_{4m+2} \subseteq SU(2)_{4m+2} \quad \exists\ 8 \text{ zesting}$$

$$\subseteq SO(2m+1)_2 \quad \exists\ 8 \text{ zesting}$$

- Categorification problems: - Grossman-Izumi 2018 new modular data
  2019 [BRW]
- Fusion rings (2019, 2020) [LPR]

THE ZESTING CONSTRUCTION

$$\mathcal{C}_g \times \mathcal{C}_h \xrightarrow{\otimes} \mathcal{C}_{gh}$$

OVERVIEW:

INPUT : $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \neq 0$ G-GRADED, PREMODULAR CATEGORY

STEP 1 : ZEST FUSION RULES $\rightarrow \tilde{\otimes}, \tilde{\alpha}$

STEP 2 : ZEST BRAID $\rightarrow \tilde{c}$

STEP 3 : ZEST TWIST $\rightarrow \tilde{\theta}$

OUTPUT : $\tilde{\mathcal{C}}$ G-GRADED PRE. MOD. CAT.

ASSOCIATIVE ZESTING

SET UP: $\mathcal{C} = \bigoplus_{a \in A} \mathcal{C}_a$ A - GRADED BRAIDED fusion CAT.

GOAL. Modify in an easy way its fusion rules to get a new fusion category

$$X_a \in \mathcal{C}_a,\ Y_b \in \mathcal{C}_b \longrightarrow X_a \overset{?}{\tilde{\otimes}} Y_b \doteq X_a \otimes Y_b \otimes \underbrace{\lambda(a,b)}_{\in\ \mathrm{INV}(\mathcal{C}_e)}$$

$\lambda : G \times G \to \mathrm{INV}(\mathcal{C}_e)$

↳ 2-cocycle cond. (normalized)

What about associativity?

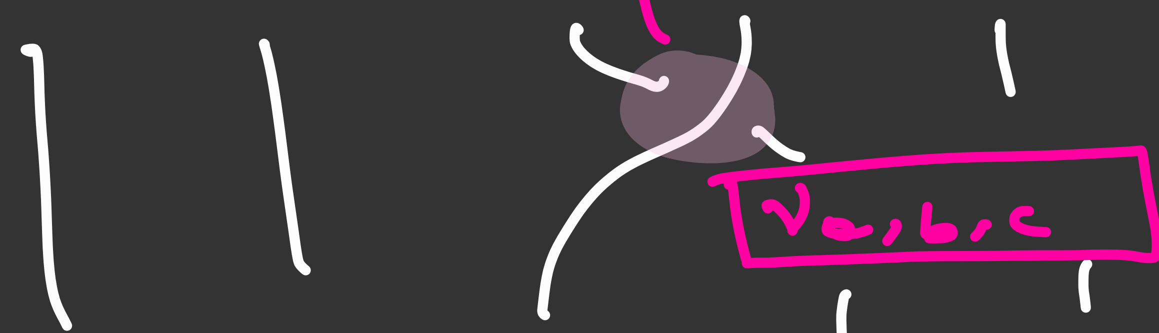
(For simplicity, assume $\mathcal{C}$ strict).

$$\tilde{\alpha}_{X_a,Y_b,Z_c} : (X_a \tilde{\otimes} Y_b) \tilde{\otimes} Z_c \xrightarrow{\sim} X_a \tilde{\otimes} (Y_b \tilde{\otimes} Z_c)$$

Braid.

$$(X_a \tilde{\otimes} Y_b) \tilde{\otimes} Z_c = X_a \; Y_b \; \lambda(a,b) \; Z_c \; \lambda(ab,c)$$

$\tilde{\alpha}_{X_a,Y_b,Z_c} \downarrow$



$$X_a \tilde{\otimes} (Y_b \tilde{\otimes} Z_c) = X_a \; Y_b \; Z_c \; \lambda(b,c) \; \lambda(a,bc)$$

$\rightsquigarrow$ 2-cocycle cond.

REMARKS :

- RANK , FPdim $\begin{cases} \text{OBJ.} \\ \text{CAT.} \end{cases}$ PRESERVED
- GRADING is THE SAME $\begin{cases} \text{GRADING GROUP} \\ \text{TRIVIAL COMP} \end{cases}$
- COHOMOLOGICAL OBSTRUCTION.
- ASSOCIATIVE ZESTING form A TORSOR OVER $H^3(A, k^\times)$.
- EXTENSION THEORY: PARTICULAR CASE of [ENO].

	PROS	CONS
GAUGINGS	• MTC for free • Witt class & c.c. preserved	• Hard to get data! ↳ fusion rules ↳ rank ↳ S,T matrix
ZESTING	• Easy to get data! ↳ rank ↳ fusion rules ↳ S,T, braid, knot & link • Easy to compute ex.	• Not more general construction • Witt class & c.c. not (neces.) preserved

QUESTIONS

COMMENTS

THANK

YOU !